

Low-temperature effective potential of the Ising model.

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Abstract

We study the low-temperature effective potential of the Ising model. We evaluate the three-point and four-point zero-momentum renormalized coupling constants that parametrize the expansion of the effective potential near the coexistence curve. These results are obtained by a constrained analysis of the ϵ -expansion that uses accurate estimates for the two-dimensional Ising model.

Keywords: Field theory, Critical phenomena, Ising model, Broken phase, Effective potential, n -point renormalized coupling constants, ϵ -expansion.

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I. INTRODUCTION

In statistical physics the effective potential corresponds to the free-energy density \mathcal{F} as a function of the order parameter, which, for spin models, is the magnetization M . The global minimum of the effective potential determines the value of the order parameter, that characterizes the phase of the model. In the high-temperature (HT) or symmetric phase the minimum is unique with $M = 0$. According to the Ginzburg-Landau theory, as the temperature decreases below the critical value, the effective potential takes a double-well shape. The order parameter does not vanish anymore and the system is in the low-temperature (LT) or broken phase. Actually in the broken phase the double-well shape is not correct because the effective potential must always be convex. In this phase it should present a flat region around the origin. For a discussion see e.g. Refs. [1,2] and references therein.

Recently many works have been devoted to the study of the effective potential — or, equivalently, of the equation of state — in the limit of small magnetization for $O(N)$ models in the HT phase. Indeed, in this case, the effective potential admits a regular expansion around $M = 0$ with coefficients that are related to the zero-momentum n -point renormalized coupling constants. Rather accurate estimates of these quantities have been obtained exploiting various field-theoretic approaches (see e.g. [3–8]) and lattice techniques (see e.g. [9–13]).

In the LT phase the effective potential shows a complex behaviour due to the Goldstone bosons and does not admit a regular expansion. The only exception is the Ising model where the symmetry is discrete and therefore the Goldstone bosons are absent. In this case, if $M_0 = \lim_{H \rightarrow 0+} M(H)$, for $M > M_0$ (i.e. for $H > 0$), the effective potential admits a regular expansion in powers of $(M - M_0)$. The coefficients of this expansion are related to the LT zero-momentum n -point renormalized coupling constants g_n . These quantities are harder to estimate than the corresponding HT couplings and indeed they are known with much less precision. At present there are some estimates for g_3 and g_4 . However the uncertainty is often very large or difficult to estimate (especially for g_4) [10,14–16]. Surprisingly one finds the same situation for the two-dimensional Ising model (see Refs. [10,15]).

In this paper we study the LT effective potential for the Ising model in two and three dimensions. We focus on the expansion in powers of $(M - M_0)$ that we parametrize in terms of w defined by

$$w^2 \equiv \lim_{T \rightarrow T_c-} \lim_{H \rightarrow 0} \frac{\chi}{M^2 \xi^d} \quad (1)$$

(where χ is the magnetic susceptibility and ξ the second-moment correlation length), and of the ratios $v_j \equiv g_j/w^{j-2}$. In two dimensions M is known exactly (see e.g. [17]) while χ and ξ can be derived using the fact that the two-point function satisfies a Painlevé differential equation in the critical limit [18]. Therefore one can easily obtain high-precision estimates of w . The ratios v_j are estimated analyzing the LT expansion (to $O(u^{23})$ where $u = e^{-4/T}$) of the free-energy on the square lattice in the presence of an external constant magnetic field [19]. In three dimensions good estimates of w^2 have been obtained from the analysis of its LT expansion [10,20] and from Monte Carlo simulations [21]. Field-theoretic calculations of w [22,23] are less precise, but perfectly consistent. In order to estimate the

constants v_j we consider their ϵ -expansion, that can be derived from the ϵ -expansion of the equation of state [24–26]. A constrained analysis of these ϵ -series using the corresponding two-dimensional results allows us to obtain estimates of v_3 and v_4 with satisfactory accuracy. It is straightforward to obtain corresponding estimates for $g_j = w^{j-2}v_j$.

The paper is organized as follows. In Sec. II we introduce our notation and give some general formulae for the LT expansion of the effective potential near the coexistence curve. In Sec. III we compute w and v_j for various values of j for the two-dimensional Ising model. In Sec. IV we present our analysis of the ϵ -expansion of v_j , and we compare our results with other approaches. In Sec. V we discuss the effective potential for the $O(N)$ model in the large- N limit. We find logarithmic terms that have not been previously predicted. We discuss some possible interpretations. In App. A we report the LT expansion of the susceptibilities that are used in the computation of v_j with $j \leq 6$ for the two-dimensional Ising model.

II. EXPANSION OF THE EFFECTIVE POTENTIAL NEAR THE COEXISTENCE CURVE

The effective potential is related to the free energy of the model. Indeed, if $M \equiv \langle \phi \rangle$ is the magnetization and H the magnetic field, one defines

$$\mathcal{F}(M) = MH - \frac{1}{V} \log Z(H), \quad (2)$$

where $Z(H)$ is the partition function and the dependence on the temperature T is always understood in the notation. We will be interested in the behaviour of $\mathcal{F}(M)$ near the coexistence curve $t \equiv T - T_c < 0$, $H = 0$. For the Ising model, if $M_0 = \lim_{H \rightarrow 0+} M(H)$, for $M > M_0$, we can expand $\mathcal{F}(M)$ in powers of $M - M_0$. Explicitly we can write

$$\mathcal{F}(M) = \mathcal{F}(M_0) + \sum_{j=2}^{\infty} \frac{1}{j!} a_j (M - M_0)^j. \quad (3)$$

It is useful to express the coefficients a_j in terms of renormalization-group invariant quantities. We therefore define a renormalized magnetization

$$\varphi^2 = \frac{\xi(t, H=0)^2 M(t, H)^2}{\chi(t, H=0)} \quad (4)$$

where ξ is the second-moment correlation length

$$\xi^2 = \frac{1}{2d} \frac{\int dx x^2 G(x)}{\int dx G(x)}, \quad (5)$$

and $G(x)$ is the connected two-point function. In terms of φ we can rewrite

$$\mathcal{F}(\varphi) = \mathcal{F}(\varphi_0) + \frac{1}{2} m^2 (\varphi - \varphi_0)^2 + \sum_{j=3} m^{d-j(d-2)/2} \frac{1}{j!} g_j (\varphi - \varphi_0)^j. \quad (6)$$

Here $m = 1/\xi$ and g_j are functions of t only. In field theory φ is nothing but the zero-momentum renormalized field. For $t \rightarrow 0-$ the quantities g_j approach universal constants (that we indicate with the same symbol) that represent the zero-momentum n -point renormalized coupling constants. An even simpler parametrization can be obtained if we introduce

$$z \equiv \frac{\varphi}{\varphi_0} = \frac{M}{M_0}. \quad (7)$$

Then we have

$$\mathcal{F}(\varphi) - \mathcal{F}(\varphi_0) = \frac{m^d}{w^2} B(z). \quad (8)$$

The function $B(z)$ has the following expansion

$$B(z) = \frac{1}{2}(z-1)^2 + \sum_{j=3} \frac{1}{j!} v_j (z-1)^j, \quad (9)$$

where

$$v_j = \frac{g_j}{w^{j-2}}. \quad (10)$$

The advantage of the expansion (9) is that its coefficients v_j are expressed only in terms of zero-momentum quantities, while in Eq. (6) the correlation length is also present.

We mention that a more natural expansion of $B(z)$ that takes into account its parity properties would be

$$B(z) = \frac{1}{4} \left[\frac{1}{2}(z^2-1)^2 + \sum_{j=3} \frac{1}{j!} \bar{v}_j (z^2-1)^j \right], \quad (11)$$

where the coefficients \bar{v}_j can be easily related to those of the expansion (9), i.e. to the constants v_j . For example $\bar{v}_3 = (3 - v_3)/2$, etc....

Since $z \propto |t|^{-\beta} M$, the equation of state can be written in the form

$$H = \left(\frac{\partial \mathcal{F}}{\partial M} \right)_t \propto |t|^{\beta\delta} \frac{\partial B(z)}{\partial z} \quad (12)$$

where we have used the hyperscaling relation $\beta(1+\delta) - d\nu = 0$. Eq. (12) can be exploited to derive $B(z)$ from the equation of state that is written in the form

$$H = M^\delta f(x) \quad (13)$$

with $x = tM^{-1/\beta}$. The scaling function $f(x)$ is usually normalized so that $f(-1) = 0$, $f(0) = 1$. The function $B(z)$ can be obtained from

$$\frac{\partial B(z)}{\partial z} = h_0 z^\delta f(x_0 z^{-1/\beta}), \quad (14)$$

where the normalization constants h_0 and x_0 are fixed by the requirement that

$$B(z) = \frac{1}{2}(z-1)^2 + O[(z-1)^3]. \quad (15)$$

Since $\beta > 0$ and the function $f(x)$ is regular at $x = 0$ and nonzero, Eq. (14) implies that $B(z) \sim z^{\delta+1}$ for $z \rightarrow \infty$.

The coefficients v_j of the expansion of $B(z)$ around $z = 1$ can be related to the LT critical limit of combinations of connected Green's functions evaluated at zero momentum

$$\chi_j = \sum_{x_2, \dots, x_j} \langle \phi(0) \phi(x_2) \dots \phi(x_{j-1}) \phi(x_j) \rangle_c. \quad (16)$$

In the LT critical limit

$$-\frac{\chi_3 M}{\chi^2} \longrightarrow v_3, \quad (17)$$

$$-\frac{\chi_4 M^2}{\chi^3} + 3 \frac{\chi_3^2 M^2}{\chi^4} \longrightarrow v_4, \quad (18)$$

$$-\frac{\chi_5 M^3}{\chi^4} + 10 \frac{\chi_4 \chi_3 M^3}{\chi^5} - 15 \frac{\chi_3^3 M^3}{\chi^6} \longrightarrow v_5, \quad (19)$$

$$-\frac{\chi_6 M^4}{\chi^5} + 15 \frac{\chi_5 \chi_3 M^4}{\chi^6} + 10 \frac{\chi_4^2 M^4}{\chi^6} - 105 \frac{\chi_4 \chi_3^2 M^4}{\chi^7} + 105 \frac{\chi_3^4 M^4}{\chi^8} \longrightarrow v_6, \quad (20)$$

etc..., where $M \equiv \chi_1$ and $\chi \equiv \chi_2$.

III. THE TWO-DIMENSIONAL ISING MODEL

In the critical limit the two-point function of the two-dimensional Ising model, both in the symmetric and broken phase, satisfies a Painlevé differential equation [18]. Therefore, w can be calculated by an appropriate numerical integration, obtaining

$$w = 0.72906.... \quad (21)$$

The coefficients v_j of the expansion of $B(z)$ are not known. Good estimates of the first few v_j can be obtained from the analysis of their LT expansion. The basic reason is that the leading correction to scaling is analytic, since the subleading exponent Δ is expected to be larger than one (see e.g. Ref. [27] and references therein). In particular the available exact calculations [18] for the square-lattice Ising model near criticality have revealed only analytic corrections to the leading power law. Therefore the traditional methods of series analysis should work well.

In order to estimate the ratios v_j we used the results of Ref. [19]. They report the expansion of the free energy in the presence of an external magnetic field to $O(u^{23})$ where $u \equiv e^{-4/T}$, for the square-lattice Ising model. Using Eqs. (17,20) one can easily obtain the corresponding expansion for v_j . In our analysis we used several types of approximants,

Padé, Dlog-Padé and first-order integral approximants¹ (for a review on the resummation techniques see for example Ref. [28]), constructed from the series of $s_j \equiv u^{-2(j-2)}v_j^{-1}$ that have the form $\sum_{i=1}^{21} c_i u^i$. Estimates of v_j are then obtained by evaluating them at $u_c = (\sqrt{2}+1)^{-2}$. In App. A we report the $O(u^{23})$ series of the zero-momentum connected correlation functions χ_j (with $j \leq 6$) we used in our analysis.

We obtained ²

$$v_3 = 33.011(6), \quad (22)$$

$$v_4 = 48.6(1.2), \quad (23)$$

$$v_5 = 7.69(2) \times 10^4, \quad (24)$$

$$v_6 \approx -2.17 \times 10^7, \quad (25)$$

$$v_7 \approx 9.9 \times 10^9, \quad (26)$$

$$v_8 \approx -6.2 \times 10^{12}. \quad (27)$$

Notice how the absolute values of v_j increase rapidly with j . Using the value of w reported in Eq. (21) we obtain:

$$g_3 = wv_3 = 24.067(4), \quad (28)$$

$$g_4 = w^2v_4 = 25.8(6). \quad (29)$$

In Ref. [10] the estimates $v_3 = 33.06(10)$ (v_3 is denoted there by R_3) and $g_4 \approx 25$ (with a large apparent uncertainty) were obtained from the analysis of the LT series published in Ref. [29] for the square, triangular and honeycomb lattices. These results are fully consistent with our analysis. The higher precision we achieved is essentially due to the longer series we considered, indeed for the square lattice Ref. [29] reports series to $O(u^{11})$. We also mention the estimate $g_3 = 23.9(2)$ reported in Ref. [15] that has been obtained by a Monte Carlo simulation for $\xi \lesssim 8$.

IV. THE THREE-DIMENSIONAL ISING MODEL

For the three-dimensional Ising model accurate estimates of w can be found in the literature. They are obtained using various approaches, from the analysis of low-

¹ Given a n th order series, we considered the following quasi-diagonal approximants: $[l/m]$ Padé and Dlog-Padé approximants with $l + m \geq n - 2$ and $l, m \geq \frac{n}{2} - 2$; $[m/l/k]$ first-order integral approximants with $m + l + k + 2 = n$ and $\lfloor (n-2)/3 \rfloor - 1 \leq m, l, k \leq \lceil (n-2)/3 \rceil + 1$.

² As estimate from each class of approximants, i.e. Padé, Dlog-Padé and first-order integral approximants, we considered the average of the values at $u = u_c$ of the non-defective approximants using all the available terms of the series. As estimate of the error we considered the square root of the variance around the estimate of the results from all the non-defective approximants listed in footnote 1. The results from Padé, Dlog-Padé and first-order integral approximants were then combined leading to the estimates shown in the text.

temperature expansions [10,20], from Monte Carlo simulations [21], and field-theoretic calculations [22,23,14]. The low-temperature expansion of w^2 can be calculated to $O(u^{21})$ on the cubic lattice using the series published in Refs. [30,31]. In the analysis of this series we used the Roskies transform [32] in order to reduce the systematic effects due to confluent singularities. Indeed, since the leading correction-to-scaling exponent $\Delta \simeq 1/2$, the systematic error due to the leading non-analytic correction may be relatively large. The analysis using Padé, Dlog-Padé and first-order integral approximants (see footnotes 1 and 2) to the Roskies transformed series of w^2 leads to the estimate $w^2 = 4.75(4)$. This is in good agreement with the estimate of w^2 that can be obtained from the results of Ref. [10], $w^2 = 4.71(5)$ [33]. The result of the Monte Carlo simulation reported in Ref. [21], $w^2 = 4.77(3)$, is consistent with these estimates. Therefore, we believe that

$$w = 2.18(1) \quad (30)$$

should be a reliable estimate. In Refs. [22,23] a perturbative approach in fixed dimension $d = 3$ is exploited to study the LT phase of the Ising model. The expansion parameter is $u \equiv 3w^2$. Its critical value is determined computing the zero of the corresponding β -function that is known to three loops. The estimate of w^2 one obtains from this expansion is consistent with Eq. (30), but less precise due to the relatively small number of known term in the β -function.

In order to calculate the first few v_j in three dimensions, we consider their ϵ -expansion. The ϵ -expansion of v_j can be derived from the ϵ -expansion of the equation of state that is known to $O(\epsilon^3)$ for the Ising model [24–26]. Using the procedure described in Sec. II, cf. Eqs. (14) and (15), one obtains

$$v_3 = 3 + \frac{3}{2}\epsilon + \frac{17}{18}\epsilon^2 + \left(\frac{989}{1944} - \frac{\lambda}{12} - \frac{\zeta(3)}{6} \right) \epsilon^3 + O(\epsilon^4), \quad (31)$$

$$v_4 = 3 + \frac{9}{2}\epsilon + \frac{43}{12}\epsilon^2 + \left(\frac{1601}{648} + \frac{\lambda}{4} - \frac{\zeta(3)}{2} \right) \epsilon^3 + O(\epsilon^4), \quad (32)$$

$$v_5 = \frac{9}{2}\epsilon + \frac{31}{3}\epsilon^2 + \left(\frac{3919}{324} - \frac{11\lambda}{4} + \frac{5\zeta(3)}{2} \right) \epsilon^3 + O(\epsilon^4), \quad (33)$$

$$v_6 = -18\epsilon - \frac{311}{6}\epsilon^2 + \left(-\frac{12167}{162} + \frac{65\lambda}{2} - 34\zeta(3) \right) \epsilon^3 + O(\epsilon^4), \quad (34)$$

etc..., where $\lambda = \psi'(1/3)/3 - 2\pi^2/9 \approx 1.17195$, and $\zeta(3) \approx 1.20206$.

Since the ϵ -expansion is asymptotic, it requires a resummation to get estimates at $d = 3$, i.e. $\epsilon = 1$. Assuming, as usual, its Borel summability, the analysis of the series is performed using the method proposed in Ref. [34], that is based on the knowledge of the large-order behaviour of the series. Given a quantity R with series

$$R(\epsilon) = \sum_{k=0} R_k \epsilon^k, \quad (35)$$

we have generated new series $R_p(\alpha, b; \epsilon)$ according to

$$R_p(\alpha, b; \epsilon) = \sum_{k=0}^p B_k(\alpha, b) \int_0^\infty dt t^b e^{-t} \frac{u(\epsilon t)^k}{[1 - u(\epsilon t)]^\alpha}, \quad (36)$$

where

$$u(x) = \frac{\sqrt{1+ax} - 1}{\sqrt{1+ax} + 1}. \quad (37)$$

Here $a = 1/3$ is the singularity of the Borel transform. The coefficients $B_k(\alpha, b)$ are determined requiring that the expansion in ϵ of $R_p(\alpha, b; \epsilon)$ coincides with the original series. For each α, b and p an estimate of R is simply given by $R_p(\alpha, b; \epsilon = 1)$. We follow Refs. [20,3] in order to derive the estimates and their uncertainty. We determine an integer value of b , b_{opt} , such that

$$R_3(\alpha, b_{\text{opt}}; \epsilon = 1) \approx R_2(\alpha, b_{\text{opt}}; \epsilon = 1) \quad (38)$$

for $\alpha < 1$. b_{opt} is the value of b such that the estimate from the series to order $O(\epsilon^3)$ is essentially identical to the estimate from the series to order $O(\epsilon^2)$. In a somewhat arbitrary way we consider as our final estimate the average of $R_p(\alpha, b; \epsilon = 1)$ with $-1 < \alpha \leq 1$ and $-2 + b_{\text{opt}} \leq b \leq 2 + b_{\text{opt}}$. The error we report is the variance of the values of $R_3(\alpha, b; \epsilon = 1)$ with $-1 < \alpha \leq 1$ and $[b_{\text{opt}}/3 - 1] \leq b \leq [4b_{\text{opt}}/3 + 1]$. A discussion of the reliability of these error estimates can be found in Refs. [20,3].

When the coefficients of the ϵ -expansion have all the same sign, as it is the case for v_j , it is convenient to consider and analyze the series of the inverse. The analysis of the $O(\epsilon^3)$ series of v_j^{-1} gives

$$v_3 = 6.05(27), \quad (39)$$

$$v_4 = 17.2(6.6). \quad (40)$$

The analysis of the ϵ -series of the coefficients v_j with $j > 4$ is extremely unstable and does not provide reliable estimates.

These results can be improved performing a constrained analysis that exploits the accurate two-dimensional estimates of v_3 and v_4 , cf. Eqs. (22) and (23). The idea is the following. Assume that exact or approximate values of the quantity at hand are known for some dimensions $d_i < 3$, with d_i belonging to the expected analytic domain in d . Then one may use the polynomial interpolation between the values $d = 4$ and $d = d_i$ as zeroth order approximation in three dimensions. If the interpolation is a good approximation one should find that the series which gives the deviations has smaller coefficients than the original one. Consequently also the errors in the resummation are reduced. The idea to constrain the ϵ -series analysis was employed in Ref. [35] to improve the estimates of the critical exponents of the Ising and self-avoiding walk models; in Ref. [36] it was used in the study of the two-point function; in Refs. [20,3] it was successfully applied to the study of the small-renormalized-field expansion of the effective potential in the symmetric phase. For quantities defined in the LT of the Ising model, the analytic domain in d should contain the value $d = 2$. Thus we can use the rather accurate two-dimensional estimates of v_3 and v_4 to constrain the analysis of the ϵ -series. In the study of the effective potential in the symmetric phase also the exact results in $d = 1, 0$ were used, because in that case the analytic domain is expected to extend up to $d = 0$ [20,3].

Assuming that the ratios v_j are analytic and sufficiently smooth in the domain $4 > d \geq 2$ (that is $0 < \epsilon \leq 2$), one may perform a linear interpolation between $d = 4$ and $d = 2$, and then analyze the series of the difference. For a generic quantity R one defines

$$\overline{R}(\epsilon) = \left[\frac{R(\epsilon) - R_{\text{ex}}(\epsilon = 2)}{(\epsilon - 2)} \right] \quad (41)$$

and a new quantity

$$R_{\text{imp}}(\epsilon) = R_{\text{ex}}(\epsilon = 2) + (\epsilon - 2)\overline{R}(\epsilon), \quad (42)$$

where $R_{\text{ex}}(\epsilon = 2)$ is the exact value of R for $\epsilon = 2$. New estimates of R for $\epsilon = 1$ can be obtained applying the resummation procedure we described above to $\overline{R}(\epsilon)$ and then computing $R_{\text{imp}}(1)$.

The constrained analysis of the ϵ -series of v_j^{-1} leads to the following results:

$$v_3 = 5.99(5 + 0), \quad (43)$$

$$v_4 = 15.8(1.4 + 0.1), \quad (44)$$

where the second error is obtained varying the two-dimensional estimate within one error bar. The new results are in good agreement with the estimates (39) and (40), but have a smaller uncertainty. Using the estimate (30) of w we can compute the three- and four-point zero-momentum renormalized coupling constants:

$$g_3 = 13.06(12), \quad (45)$$

$$g_4 = 75(7). \quad (46)$$

The errors of g_3 and g_4 are calculated considering the errors of w , v_3 , and v_4 as independent.

Let us compare our results with available estimates obtained using other methods. Table I presents a summary of all the available (as far as we know) estimates of v_3 , g_3 and g_4 . Field-theoretic estimates of v_3 are obtained in Refs. [4,5] using the parametric representation of the equation of state (in Table I we refer to this approach by PR). In this approach the ϵ -expansion and the $d = 3$ g -expansion (i.e. expansion in powers of the HT zero-momentum four-point renormalized coupling) are used to estimate the first few coefficients of the expansion in powers of θ of the function $h(\theta)$ characterizing the parametric representation of the equation of state. From $h(\theta)$ one can derive many universal ratios of quantities defined at zero momentum such as v_j . Ref. [10] presents an analysis of the low-temperature expansion on the cubic, b.c.c. and f.c.c. lattices, obtaining $v_3 = 6.47(20)$, and $g_4 \approx 85$ with a large apparent uncertainty. These estimates are in good agreement with our results (43) and (46). We also mention Ref. [14] where the effective potential in the broken phase is determined from an approximate solution of the exact renormalization-group equations. The resulting estimates are $w^2 = 5.55$ and $g_3 = 15.24$. It is very hard to obtain good estimates by Monte Carlo simulations. Ref. [15] reports some results obtained with a Monte Carlo simulation with $\xi \lesssim 8$. They obtain $g_3 = 16.0(5)$, and $g_4 - 3g_3^2 \approx 750$ (the precision of the data does not allow us to extract an estimate of g_4). Ref. [16] reports estimates of g_3 and g_4 that are obtained from the probability distribution of the average magnetization,

whose numerical data are analyzed by assuming a sixth-order polynomial approximation of the effective potential. In Table I we report the data of Ref. [16] corresponding to the largest correlation length $\xi \simeq 7$ and lattice 74^3 (taken from the 5th column of Table 2 as suggested by the author). While for g_3 there is a substantial agreement, the estimate of g_4 turns out to be quite larger than ours.

V. THE EFFECTIVE POTENTIAL IN THE LARGE- N LIMIT

The physics of the broken phase of $O(N)$ models with $N > 1$ is very different from that of the Ising model, because of the presence of Goldstone modes. In this case the effective potential $\mathcal{F}(\varphi)$ is not analytic for $\varphi \rightarrow \varphi_0$. General renormalization-group arguments predict

$$\mathcal{F}(\varphi) - \mathcal{F}(\varphi_0) \approx c (\varphi^2 - \varphi_0^2)^\rho, \quad (47)$$

where c is a constant and ρ a new exponent that is conjectured to be given exactly by $\rho = d/(d-2)$. The conjecture is based on the following argument. The exponent ρ is related to the behaviour of the longitudinal susceptibility χ_L along the coexistence curve. From Eq. (47) it is easy to derive

$$\chi_L = \frac{\partial M}{\partial H} \sim H^{(2-\rho)/(\rho-1)}, \quad (48)$$

for $H \rightarrow 0$. On the other hand the singularity of χ_L for $H \rightarrow 0$ is governed by the zero-temperature infrared-stable fixed point [37–39]. This leads to the exact prediction

$$\chi_L \sim H^{d/2-2}, \quad (49)$$

and therefore $\rho = d/(d-2)$. The asymptotic behaviour (47) has been checked in the large- N limit to leading and subleading order [37]. At leading order Eq. (47) holds for all φ and not only in the limit $\varphi \rightarrow \varphi_0$.

The nature of the corrections to the behaviour (47) is less clear. Setting $x = tM^{-1/\beta}$ and $y = HM^{-\delta}$, the conjecture is that $1+x$ has the form of a double expansion in powers of y and $y^{(d-2)/2}$ near the coexistence curve [40,41,39], i.e. for $y \rightarrow 0$

$$1+x = c_1 y + c_2 y^{1-\epsilon/2} + d_1 y^2 + d_2 y^{2-\epsilon/2} + d_3 y^{2-\epsilon} + \dots \quad (50)$$

where $\epsilon = 4-d$. Here we assume, as usual, that $x = -1$ corresponds to the coexistence curve. With this normalization $z = (-x)^{-\beta}$. Thus, for $x \rightarrow -1$, $1+x \approx (z-1)/\beta$ with corrections that are analytic in $(z-1)$. Therefore, also $z-1$ should have an expansion of the same form.

Notice that this expansion predicts that, in three dimensions, $z-1$ has an expansion that is analytic in powers of $y^{1/2}$. As a consequence, the effective potential would be analytic in $z-1$, with $B(z) \sim (z-1)^3$ for $z \rightarrow 1$.

These ideas can be verified in the large- N limit using the equation of state to order $O(1/N)$ (next-to-leading in $1/N$) reported in Ref. [37]. Setting $\omega = 1+x$, we find that, in generic dimension d with $2 < d < 4$, the function $f(x)$ defined in Eq. (13) has an expansion of the form

$$f(x) = \omega^{2/(d-2)} \left[1 + \frac{1}{N} \left(\sum_{i=0}^{\infty} c_i \omega^i + \sum_{n=-1}^{\infty} \sum_{m=1}^{\infty} c_{mn} \omega^{2m/(d-2)+n} \right) + O\left(\frac{1}{N^2}\right) \right], \quad (51)$$

which is consistent with the expansion (50). Generically, the coefficients c_{mn} are singular for those values of d such that $2m/(d-2)$ is an integer, i.e. for

$$2 < d = 2 + \frac{2m}{n} < 4, \quad \text{for } 0 < m < n, \quad m, n \in \mathbb{N}. \quad (52)$$

However, in these cases, also the coefficients c_i are singular. A careful analysis shows that for these special values of d logarithmic terms should appear. We have verified analytically their presence for $d = 4 - 2/n$, that corresponds to $m = n - 1$ in Eq. (52). Explicitly in three dimensions ($n = 2$) we find

$$f(x) = \omega^2 \left[1 + \frac{1}{N} (f_1(\omega) + \log \omega f_2(\omega)) + O(N^{-2}) \right]. \quad (53)$$

The functions $f_1(\omega)$ and $f_2(\omega)$ have a regular expansion in powers of ω . In particular

$$f_2(\omega) = -\frac{23}{6}\omega^2 - \frac{1}{16}(128 + \pi^2)\omega^3 + \frac{1}{160}(2000 + 17\pi^2)\omega^4 + O(\omega^5). \quad (54)$$

Notice the absence of a linear term in $f_2(\omega)$. This is due to the fact that $c_{1,-1}$ and c_1 in Eq. (51) are not singular for $d \rightarrow 3$.

The logarithmic terms we find are not predicted by Eq. (50). In particular Eq. (53) is incompatible with Eq. (50), that, as we already remarked, predicts an analytic expansion in powers of ω . The presence of logarithms in the expansion for values of d arbitrarily near 4 casts some doubts on the asymptotic behavior (50) that has been derived essentially from an ϵ -expansion analysis.

What do we learn for the behaviour of the critical effective potential near the coexistence curve for finite values of N ? A possible interpretation of the results is that the expansion (53) holds for all values of N and thus Eq. (50) is correct apart from logarithms that are present for some special values of d . The reason of their appearance is however unclear. Nevertheless, it does not necessarily contradict the conjecture that the behavior near the coexistence curve is controlled by the zero-temperature infrared-stable Gaussian fixed point. In this case logarithms would not be unexpected, as they usually appear in the reduced temperature asymptotic expansion around Gaussian fixed points (see e.g. Ref. [42]). Anyway, since they appear also for dimensions $d > 3$, they cannot be related to the presence of marginal operators, say $(\varphi^2)^3$ in three dimensions.

There is another possible interpretation of the result (53) that we mention for completeness. In the large- N expansion, logarithmic terms are usually the signal of the presence of N -dependent critical exponents. For instance consider a term of the form

$$A(N)\omega^{\sigma(N)} + B(N)\omega^m, \quad (55)$$

with $\sigma(N) = m + \sigma_1/N + O(N^{-2})$ and m integer. For large values of N , expanding $A(N) = A_0 + A_1/N + O(N^{-2})$ and analogously $B(N)$, we obtain

$$(A_0 + B_0)\omega^m + \frac{1}{N} [(A_1 + B_1)\omega^m + \sigma_1 A_0 \omega^m \log \omega] + O(N^{-2}). \quad (56)$$

This expansion would reproduce the behaviour (53), provided $A_0 + B_0 = 0$. In generic dimension Eq. (55) would be compatible with the expansion (51) if

$$\sigma = \frac{2k}{d-2} + m - 2k + O(N^{-1}), \quad (57)$$

for some integer $k \leq m/2$. Notice that the expansion in powers of $1/N$ of the coefficients $A(N)$ and $B(N)$ would be discontinuous in d , since $A(N)$ and $B(N)$ would be of order $1/N$ in generic dimension and of order 1 whenever $\sigma(\infty)$ is an integer. A similar phenomenon has been found for the HT specific heat [43,44], and in the expansion of the Callan-Symanzik β -function near the critical point [20]. This interpretation of the results is quite natural in the framework of the large- N expansion. However it is not clear which exponents should appear. In particular, if the singular behaviour is controlled by the zero-temperature infrared-stable Gaussian fixed point, we would not expect any N -dependent exponent. We think that these questions deserve further investigation.

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APPENDIX A: TWO-DIMENSIONAL LT SERIES

In this appendix we report the LT expansions of the connected correlation functions χ_j for the square-lattice Ising model with nearest-neighbor interactions. They are derived from the expansion of the free energy in the presence of an external magnetic field [19]. This series reproduces the exact results for the free energy and the magnetization in the absence of a magnetic field [45,17]

$$F = \int_0^\pi \frac{d\theta}{2\pi} \log \left\{ \frac{(1+u)^2}{2u} + \frac{1}{2u} [(1+u)^4 - 16u(1-u)^2 \sin^2 \theta]^{1/2} \right\}, \quad (A1)$$

$$M = \left[1 - \frac{16u^2}{(1-u)^4} \right]^{1/8}, \quad (A2)$$

where $u = e^{-4/T}$. This check is particularly effective as all the terms of the expansion contribute in the limit $H \rightarrow 0$.

The expressions for the susceptibilities, cf. Eq. (16), up to terms of order $O(u^{24})$ are the following:

$$\begin{aligned} \chi = & u^2(4 + 32u + 240u^2 + 1664u^3 + 11164u^4 + 73184u^5 + 472064u^6 + 3008032u^7 \\ & + 18985364u^8 + 118909888u^9 + 740066448u^{10} + 4581660832u^{11} \\ & + 28237063308u^{12} + 173353630848u^{13} + 1060674765568u^{14} \end{aligned}$$

$$\begin{aligned}
& +6470624695296u^{15} + 39370663086596u^{16} + 238993166711328u^{17} \\
& +1447734754083760u^{18} + 8753312020985216u^{19} + 52833859249062236u^{20} \\
& +318401517346022240u^{21})
\end{aligned} \tag{A3}$$

$$\begin{aligned}
\chi_3 = & u^2(-8 - 128u - 1648u^2 - 17216u^3 - 161608u^4 - 1407040u^5 - 11607872u^6 \\
& -91914944u^7 - 704714056u^8 - 5264048832u^9 - 38484337136u^{10} \\
& -276313973760u^{11} - 1953615567368u^{12} - 13630566361728u^{13} \\
& -94009039294720u^{14} - 641820330787712u^{15} - 4342587216599624u^{16} \\
& -29147168513086720u^{17} - 194228811717309808u^{18} - 1285893394538102720u^{19} \\
& -8463157853682576776u^{20} - 55401729118133053376u^{21})
\end{aligned} \tag{A4}$$

$$\begin{aligned}
\chi_4 = & u^2(16 + 512u + 11232u^2 + 174464u^3 + 2268016u^4 + 26007680u^5 \\
& +272557376u^6 + 2666887552u^7 + 24721765328u^8 + 219362198656u^9 \\
& +1877373570336u^{10} + 15586471370752u^{11} + 126094785804720u^{12} \\
& +997562812048896u^{13} + 7739640438305152u^{14} + 59027832457279488u^{15} \\
& +443396861865190928u^{16} + 3285761490274027008u^{17} + 24053962791473093728u^{18} \\
& +174163313822385408896u^{19} + 1248496572536156377712u^{20} \\
& +8868728488366239392384u^{21})
\end{aligned} \tag{A5}$$

$$\begin{aligned}
\chi_5 = & u^2(-32 - 2048u - 76864u^2 - 1763072u^3 - 31561504u^4 - 474188032u^5 \\
& -6284096768u^6 - 75680753408u^7 - 845304372256u^8 - 8883063406848u^9 \\
& -88761722644544u^{10} - 850124864318976u^{11} - 7853055915373856u^{12} \\
& -70313539072037376u^{13} - 612653077144892416u^{14} - 5211746985723756032u^{15} \\
& -43403126238486164768u^{16} - 354661581988366266880u^{17} \\
& -2849039732654299216960u^{18} - 22536604900114543443200u^{19} \\
& -175792207497016658107424u^{20} - 1353835890235723182430976u^{21})
\end{aligned} \tag{A6}$$

$$\begin{aligned}
\chi_6 = & u^2(64 + 8192u + 531840u^2 + 17944064u^3 + 440964544u^4 + 8650801664u^5 \\
& +144516779264u^6 + 2136034381312u^7 + 28672334567744u^8 \\
& +356014396523008u^9 + 4144706261854848u^{10} + 45707724350236672u^{11} \\
& +481293469566545088u^{12} + 4869609919772817408u^{13} + 47583052907024977408u^{14} \\
& +450914521123508084736u^{15} + 4158352425040048310336u^{16} \\
& +37427697740234745151488u^{17} + 329592281476942345924480u^{18} \\
& +2845676740202773130026496u^{19} + 24132653400815900244380096u^{20} \\
& +201334455541709269472268800u^{21})
\end{aligned} \tag{A7}$$

We should note that the expansion of χ does not reproduce the expression reported in Ref. [19]. The last two terms are slightly different:

$$\chi(\text{our}) - \chi(\text{Ref. [19]}) = 52u^{22} + 872u^{23} \tag{A8}$$

The discrepancy is probably due to a misprint, since their general series (in $\mu = e^{-2H}$ and u) correctly reproduces Eqs. (A1,A2).

REFERENCES

- [1] L. O’Raifeartaigh, A. Wipf, and H. Yoneyama, Nucl. Phys. **B271** (1986) 653.
- [2] H. Mukaida and Y. Shimada, Nucl. Phys. **B479** (1996) 663.
- [3] A. Pelissetto and E. Vicari, Nucl. Phys. **B522** [FS] (1998) 605.
- [4] R. Guida and J. Zinn-Justin, Nucl. Phys. **B489** (1997) 626.
- [5] R. Guida and J. Zinn-Justin, cond-mat/9803240.
- [6] A. I. Sokolov, Fizika Tverdogo Tela **40** (1998) 1284.
- [7] N. Tetradis and C. Wetterich, Nucl. Phys. **B422** (1994) 541.
- [8] T. Morris, Nucl. Phys. **B495** (1997) 477.
- [9] P. Butera and N. Comi, Phys. Rev. **E55** (1997) 6391.
- [10] S. Zinn, S.-N. Lai, and M. E. Fisher, Phys. Rev. **E54** (1996) 1176.
- [11] M. M. Tsy-pin, Phys. Rev. Lett. **73** (1994) 2015.
- [12] G. A. Baker Jr. and N. Kawashima, J. Phys. **A29** (1996) 7183.
- [13] J.-K. Kim and D. P. Landau, Nucl. Phys. B (Proc. Suppl.) **53** (1997) 706.
- [14] J. Berges, N. Tetradis and C. Wetterich, Phys. Rev. Lett. **77** (1996) 873.
- [15] J. K. Kim and A. Patrascioiu, Phys. Rev. **D47** (1993) 2588.
- [16] M. M. Tsy-pin, Phys. Rev. **B55** (1997) 8911.
- [17] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, (Academic Press, London, 1982).
- [18] T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Phys. Rev. **B13** (1976) 316.
- [19] R.J. Baxter and I. G. Enting, J. Stat. Phys. **21**, (1979) 103.
- [20] A. Pelissetto and E. Vicari, Nucl. Phys. **B519** [FS] (1998) 626.
- [21] M. Caselle and M. Hasenbusch, J. Phys. **A30** (1997) 4963.
- [22] G. Münster and J. Heitger, Nucl. Phys. **B424** (1994) 582.
- [23] C. Gutfeld, J. Küster, and G. Münster, Nucl. Phys. **B479** (1996) 654.
- [24] E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. Lett. **29** (1972) 591; Phys. Rev. **B7** (1973) 232.
- [25] D. J. Wallace and R. P. K. Zia, J. Phys. **C7** (1974) 3480.
- [26] J. F. Nicoll and P. C. Albright, Phys. Rev. **B31** (1985) 4576.
- [27] M. Barma and M. E. Fisher, Phys. Rev. **B31** (1985) 5954.
- [28] A. J. Guttmann, *Phase Transitions and Critical Phenomena*, vol. 13, C. Domb and J. Lebowitz eds. (Academic Press, New York, 1989).
- [29] M. F. Sykes, D. S. Gaunt, J. L. Martin, S. R. Mattingly, and J. W. Essam, J. Math. Phys. **14** (1973) 1071.
- [30] H. Arisue and K. Tabata, Nucl. Phys. **B435** (1995) 555.
- [31] C. Vohwinkel, Phys. Lett. **B301** (1993) 208.
- [32] R. Z. Roskies, Phys. Rev. **B24** (1981) 5305.
- [33] M. E. Fisher, private communication.
- [34] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. **39** (1977) 95; Phys. Rev. **B21** (1980) 3976.
- [35] J. C. Le Guillou and J. Zinn-Justin, J. Physique **48** (1987) 19.
- [36] M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. **E57** (1998) 184.
- [37] E. Brézin and D. J. Wallace, Phys. Rev. **B7** (1973) 1967.
- [38] E. Brézin and J. Zinn-Justin, Phys. Rev. **B14** (1976) 3110.

- [39] I. D. Lawrie, J. Phys. **A14** (1981) 2489.
- [40] D. J. Wallace and R. P. K. Zia, Phys. Rev. **B12** (1975) 5340.
- [41] L. Schäfer and H. Horner, Z. Phys. **B29** (1978) 251.
- [42] C. Bagnuls and C. Bervillier, Phys. Rev. **B 32** (1985) 7209.
- [43] R. Abe and S. Hikami, Prog. Theor. Phys. **51** (1974) 1041.
- [44] R. Abe and S. Hikami, Prog. Theor. Phys. **54** (1975) 325.
- [45] L. Onsager, Nuovo Cimento (Suppl.) **6** (1949) 61.

TABLES

TABLE I. We report the available results for v_3 , g_3 and g_4 . The estimates of g_3 marked by an asterisk have been derived by us using the value of v_3 of the corresponding line and the estimate (30) of w .

	v_3	g_3	g_4
ϵ -exp. [this paper]	5.99(5)	13.06(12)	75(7)
ϵ -exp. PR [5]	6.07(17)	*13.2(4)	
$d = 3$ -exp. PR [5]	6.08(6)	*13.25(14)	
ERG [14]	6.47	15.24	
LT [10]	6.47(20)	13.9(4)	≈ 85
MC [16]		13.6(5)	108(7)